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# ON OPTIMAL THICKNESS OF A CYLINDRICAL SHELL LOADED BY EXTERNAL PRESSURE

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The problem of an optimal, from the weight viewpoint, thickness distribution law along the length of a cylindrical shell loaded by axisymmetric external pressure is examined when collapse occurs because of buckling. The apparatus of the generalized maximum principle is used for the solution [1].

**1.** The problem is formulated in the terminology of the theory of optimal processes. The state of the shell during loading is given by the phase coordinates $\varphi_j$ $(j = 1, 2, \ldots, 6)$ at each instant $\alpha$, where $\alpha$ is the dimensionless length coordinate. A change in the phase coordinates, as $\alpha$ changes, corresponds to shell motion. This process can be controlled by changing the shell thickness $\delta(\alpha)$. The highest derivative $\delta^n(\alpha)$ [2] in the motion (stability) equation is taken as the control function, and the functions $\delta(\alpha), \ldots, \delta^{n-1}(\alpha)$ as the phase coordinates. The problem is to seek a function $\delta(\alpha)$ satisfying the stability equations, as well as boundary conditions and constraints, such that the minimum of the quantity

$$J = \int_0^{L/R} \delta(\alpha)\, d\alpha$$

would be achieved. Here $R$ and $L$ are the shell radius and length, respectively.

Constraints are imposed from structural or engineering considerations, as well as from the strength condition $\delta(\alpha) \geqslant \delta_{\min}$ and the additional condition associated with the selected model of shell analysis $\delta^n(\alpha) \leqslant a$. An optimal shell is sought in the class of admissible shells, which can be computed by using the Kirchhoff-Love hypothesis. Hence, the stability equations of classical shell theory are taken as the trajectory equations, and a constraint from the condition [3]

$$\frac{1}{R}\frac{d\delta(\alpha)}{d\alpha} \leqslant \frac{\delta_{\max}}{R} \tag{1.1}$$

is imposed on the quantity $\delta^n(\alpha)$ In investigating other classes of shells, reinforced, say, appropriate constraints and stability equations must be given.

**2.** The stability equations obtained by using the hypothesis of semi-membrane theory for shells with an arbitrary change in nonsymmetric thickness relative to the coordinate surface, are

$$\Delta[D(\alpha,\beta)\Delta\varphi_*(\alpha,\beta)] + R^2\frac{\partial^2}{\partial\alpha^2}\left[\frac{E\delta(\alpha,\beta)}{1-\mu^2}\frac{\partial^2\varphi_*(\alpha,\beta)}{\partial\alpha^2}\right] + R\Delta\left[B(\alpha,\beta)\frac{\partial^2\varphi_*(\alpha,\beta)}{\partial\alpha^2}\right] +$$
$$+ R\frac{\partial^2}{\partial\alpha^2}[B(\alpha,\beta)\Delta\varphi_*(\alpha,\beta)] + q(\alpha)R\frac{\partial^2}{\partial\beta^2}\Delta\varphi_*(\alpha,\beta) = 0$$
$$\Delta = \frac{\partial^2}{\partial\beta^2}\left(\frac{\partial^2}{\partial\beta^2}+1\right), \quad B(\alpha,\beta) = \frac{E}{2(1-\mu^2)}[\delta^2(\alpha,\beta) - \delta_{\min}\delta(\alpha,\beta)] \qquad (2.1)$$
$$D(\alpha,\beta) = \frac{E}{3(1-\mu^2)}\left[\delta^3(\alpha,\beta) - \frac{3}{2}\delta_{\min}\delta^2(\alpha,\beta) + \frac{3}{4}\delta_{\min}^2\delta(\alpha,\beta)\right]$$

Here $\beta$ is the arc coordinate, $q(\alpha)$ is the external load, $E$ is the Young modulus, and $\mu$ is the Poisson ratio. The thickness is assumed constant in the arc direction. After separation of variables by substituting $\varphi_*(\alpha,\beta) = \varphi(\alpha)\cos k\beta$, Eq. (2.1) is written as a system of first order differential equations (in the phase coordinates)

$$\varphi_1^{\cdot} = \varphi_2, \quad \varphi_2^{\cdot} = \varphi_3, \quad \varphi_3^{\cdot} = \varphi_4, \quad \varphi_4^{\cdot} = F(\alpha), \quad \varphi_5^{\cdot} = \varphi_6, \quad \varphi_6^{\cdot} = u \qquad (2.2)$$

$$F(\alpha) = \frac{1}{\varphi_5}\sum_{j=1}^{4} F_j(\alpha)\varphi_j$$
$$F_1(\alpha) = \frac{qR}{\delta_0 E}(1-\mu^2)k^4(k^2-1) - \frac{\lambda^2}{12}(4\varphi_5^3 - 6\varphi_5^2 + 3\varphi_5) -$$
$$-\frac{\lambda}{2}(2\varphi_6^2 - 2\varphi_5 u - u), \quad F_2(\alpha) = -\lambda(2\varphi_5\varphi_6 - \varphi_6)$$
$$F_3(\alpha) = -u - \lambda(\varphi_5^2 - \varphi_5), \qquad F_4(\alpha) = -2\varphi_6$$
$$\delta(\alpha) = \delta_0\varphi_5, \qquad \lambda = \frac{\delta_0}{R}k^2(k^2-1)$$

The boundary conditions for the solution of the system (2.2) are written from the hinged-support conditions on the shell edges

$$\varphi_1 = \varphi_3 = 0, \quad \varphi_2, \varphi_4 - \text{ free for } \alpha = 0\,(S_0)$$
$$\varphi_1 = \varphi_3 = 0, \quad \varphi_2, \varphi_4 - \text{ free for } \alpha = L/R\,(S_1) \qquad (2.3)$$

The function $\varphi_5$ is given as unity on the edge. The problem is to seek a control $u = u(\alpha)$ which optimally (in the sense of minimum $I$) transfers the object under consideration from the set $S_0$ $(\alpha = 0)$ into the set $S_1$ $(\alpha = L/R)$ and satisfies the system (2.2) and the constraints

$$u \leqslant \gamma, \qquad \varphi_5 \geqslant \delta_{\min}/\delta_0 \qquad (2.4)$$

where $\gamma$ is found from the condition (1.1). The solution of the problem posed can be obtained by using the theorem called the generalized maximum principle [1].

**3.** The necessary condition of optimality in the presence of constraints for each portion of the trajectory which does not contain points inside where jump conditions are satisfied, is for the problem formulated

$$\frac{d\varphi_j}{d\alpha}=\frac{\partial H}{\partial \psi_j}=F_j^*(\alpha,\varphi,u),\qquad \frac{d\psi_j}{d\alpha}=-\frac{\partial H}{\partial \varphi_j}$$
$$H=H(\psi,\alpha,u,\varphi)=\Sigma\psi_j F_j^*(\alpha),\qquad F_j^*(\alpha)=\frac{F_j(\alpha)}{\varphi_5} \tag{3.1}$$
$$H(\psi,\alpha,u,\varphi)=M(\psi,\alpha,\varphi),\qquad \psi_0(\alpha)=\text{const}\leqslant 0$$
$$M(\psi,\alpha,\varphi)=\sup H(\psi,\alpha,u,\varphi)$$

The transversality conditions

$$M_N=\sum_{t_N=1}^{S_N}\nu_{t_N}\frac{\partial}{\partial\alpha}\xi_{t_N}(\alpha,\varphi_N)\,|_{\alpha_N} \tag{3.2}$$
$$\psi(\alpha_N)=\sum_{t_N=1}^{S_N}\nu_{t_N}\operatorname{grad}\xi_{t_N}(\alpha_N,\varphi)\,|_{\varphi_N}$$

must be satisfied at both ends of the trajectory $(N=0,1)$, where $\nu_{t_N}, t_N, S_N$ are arbitrary constants, $\xi_{t_N}(\alpha,\varphi)=0$ are systems of equations describing the manifolds $S_0$ and $S_1$ in space. Hence, $\psi_2=\psi_4=\psi_6=0$, $\psi_1,\psi_3$ are arbitrary for $\alpha_0=0$ and $\psi_2=\psi_4=\psi_6=0$, $\psi_1,\psi_3,\psi_5$ are arbitrary for $\alpha_1=L/R$.

We obtain for the system (2.2)

$$\frac{d\psi_1}{d\alpha}=-F_1(\alpha)\frac{\psi_4}{\varphi_5},\qquad \frac{d\psi_2}{d\alpha}=-\psi_1+F_2(\alpha)\frac{\psi_4}{\varphi_5}$$
$$\frac{d\psi_3}{d\alpha}=-\psi_2+F_3(\alpha)\frac{\psi_4}{\varphi_5},\qquad \frac{d\psi_4}{d\alpha}=-\psi_3+F_4(\alpha)\frac{\psi_4}{\varphi_5} \tag{3.3}$$
$$\frac{d\psi_5}{d\alpha}=-\psi_0-\Big[\varphi_5L_1(\alpha)-\sum_j F_j(\alpha)\Big]\frac{\psi_4}{\varphi_5^2},\qquad \frac{d\psi_6}{d\alpha}=-\psi_5-L_2(\alpha)\frac{\psi_4}{\varphi_5}$$
$$L_1(\alpha)=-\varphi_1[\lambda^2(\varphi_5^2-\varphi_5+{}^1\!/_4)+\lambda u]-\lambda[2\varphi_6\varphi_2+2\varphi_5\varphi_3-\varphi_3)$$
$$L_2(\alpha)=-\lambda(2\varphi_6\varphi_1+2\varphi_5\varphi_2-\varphi_2)-2\varphi_4$$
$$H=\psi_0\varphi_5+\psi_1\varphi_2+\psi_2\varphi_3+\psi_3\varphi_4+\psi_4F(\alpha)+\psi_5\varphi_6+\psi_6u$$

Let us write the constraints (2.4) in the form of the relationships

$$\Phi_1(\alpha,\varphi,u)=u-\gamma\leqslant 0,\qquad \Phi_2(\alpha,\varphi,u)=-u-\gamma\leqslant 0 \tag{3.4}$$
$$x_3(\alpha,\varphi)=\delta_{\min}/\delta_0-\varphi_5\leqslant 0 \tag{3.5}$$

The relationships (3.4) and (3.5) isolate some admissible domain whose boundary is formed by the hypersurfaces

$$\Phi_\nu(\alpha,\varphi,u)=0\qquad (\nu=1,2) \tag{3.6}$$
$$x_3(\alpha,\varphi)=0 \tag{3.7}$$

The optimal trajectory $\varphi(\alpha)$ can contain pieces belonging to the boundary surface (3.7). It is customary to call any of the edges of a simply-connected piece belonging to the boundary surface $x_3(\alpha,\varphi)=0$ a splice point [2]; the edge which is the beginning is the entrance point $(\tau_0,\varphi(\tau_0))$, and which is the end is the point of descent $(\tau_1,\varphi(\tau_1))$. It can turn out that a piece consists of a single splice point, the point of reflection of the trajectory $\varphi(\alpha)$ from the boundary $x_3(\alpha,\varphi)=0$ Such a point will be simultaneously an entrance point and a point of descent.

The domain $u$ is defined everywhere by the relationships (2.4), to which at the points

$$d^r x_3/d\alpha^r=x_3^r(\alpha,\varphi)=0\qquad (r=0,1,\ldots,P_{\nu-1}) \tag{3.8}$$

must be added the relationship

$$\frac{d^{P_\nu}}{d\alpha^{P_\nu}} x_3(\alpha, \varphi) = \Phi_3(\alpha, \varphi, u) \leqslant 0 \tag{3.9}$$

In the case under consideration (3.8) and (3.9) have the form

$$x_3^1 = \varphi_6 = 0, \quad x_3^2 = \Phi_3(\alpha, \varphi, u) = -u \leqslant 0 \quad (\tau_0 \leqslant \alpha \leqslant \tau_1) \tag{3.10}$$

Conditions (3.8) must be satisfied at the entrance points $(\tau_0, \varphi(\tau_0))$ of the admissible trajectory on any boundary hypersurface, whereupon the jump condition

$$M[\psi(\alpha^*+0), \alpha^*, \varphi(\alpha^*)] = M[\psi(\alpha^*-0), \alpha^*, \varphi(\alpha^*)] - $$
$$- \sum_{r=0}^{r_1} \varkappa_\nu^r \frac{\partial}{\partial\alpha} x_\nu^r(\alpha, \varphi)|_{\alpha^*} \tag{3.11}$$

$$\psi(\alpha^*+0) = \psi(\alpha^*-0) - \sum_{r=0}^{r_1} \varkappa_\nu^r \operatorname{grad} x_\nu^r(\alpha^*, \varphi)|_{\varphi(\alpha^*)}$$

$$(\tau_0 \leqslant \alpha^* \leqslant \tau_1)$$

$$r_1 = \begin{cases} P_\nu - 1, & \tau_\nu \neq \tau_1 \\ P_\nu' - 1, & \tau_0 = \tau_1 \end{cases}$$

must be satisfied on each simply-connected piece, where $\varkappa_\nu^r$ are arbitrary constants. It is sufficient to satisfy the jump condition once at any point $\alpha^*$ of the segment $[\tau_0, \tau_1]$. Because of the jump condition (3.11), the functions $\psi_1$, $\psi_2$, $\psi_3$ and $\psi_4$ are continuous and

$$\psi_5(\tau_1+0) = \psi_5(\tau_1-0) + \varkappa_3^0$$
$$\psi_6(r_1+0) = \psi_6(\tau_1-0) + \varkappa_3^1$$
$$M(\tau_1+0) - M(\tau_1-0) = (\psi_5^+ - \psi_5^-)\varphi_6(\tau_1) - (\psi_6^+ - \psi_6^-) \times$$
$$\times (u^+ - u^-) = \varkappa_3^0 \varphi_6(\tau_1) + \varkappa_3^1 u$$

Hence $\varkappa_3^1 = 0$, i.e. the function $\psi_6$ is also continuous. We obtain from the condition of maximum of the function $H$ of the variable $u(\alpha)$

$$u = \gamma \operatorname{sign} K(\alpha, \varphi) \tag{3.12}$$
$$K(\alpha, \varphi) = \frac{\psi_4}{2\varphi_5}[\lambda(1-2\varphi_5)\varphi_1 - 2\varphi_3] + \psi_6$$

Therefore, we have two systems of sixth order differential equations (2.2) and (3.3) and a system of twelve boundary conditions (2.3) and (3.2), which in combination with condition (3.12) and (3.10) for the sections of the trajectory lying on the boundary surface yield the necessary conditions for solution of the problem posed.

**4.** Let us use the notation

$$u = u(\alpha) = \gamma \operatorname{sign} K(\alpha, \varphi) = u_0 \qquad (0 \leqslant \alpha \leqslant \alpha_1) \tag{4.1}$$

where $\alpha_1$ is the first point of "switching" the control corresponding to the condition $K(\alpha_1, \varphi) = 0$. Let us consider that there are $n$ roots $K(\alpha, \varphi)$ on the section $0 \leqslant \alpha \leqslant L/R$ correspondingly, $n$ points for switching the control $u(\alpha)$. Integrating the fifth and sixth equations of the system (2.2) we obtain

$$\varphi_6 = u_0\alpha + C_1, \quad \varphi_5 = {}^1/_2 u_0\alpha^2 + C_1\alpha + C_2 \qquad (0 \leqslant \alpha \leqslant \alpha_1)$$

$$\varphi_6 = -u_0\alpha + 2u_0\alpha_1 + C_1$$

$$\varphi_5 = -{}^1/_2 u_0\alpha^2 + (2u_0\alpha_1 + C_1)\alpha + C_2 - u_0\alpha_1^2$$

$$(\alpha_1 \leqslant \alpha \leqslant \alpha_2)$$

$$\varphi_6 = (-1)^n u_0\alpha + 2v_0\sum_{i=1}^{n}(-1)^{i+1}\alpha_i + C_1$$

$$\varphi_5 = \frac{1}{2}(-1)^n u_0\alpha^2 + \left[2u_0\sum_{i=1}^{n}(-1)^{i+1}\alpha_i + C_1\right]\alpha + u_0\sum_{i=1}^{n}\alpha_i^2(-1)^i + C_2 \quad (4.2)$$

$$(\alpha_n \leqslant \alpha \leqslant L/R)$$

$$C_1 = \varphi_6(0), \qquad C_2 = \varphi_5(0)$$

After substituting the values of $\varphi_5(\alpha)$ and $\varphi_6(\alpha)$ in the first four equations of the systems (2.2) and (3.3), linear systems of equations with variable coefficients are obtained. The solution of the systems reduces to seeking the eigenvalue $\delta_0$ corresponding to the nontrivial solution.

The solution of the system (2.2) and (3.3) is represented in a form satisfying the system of boundary conditions

$$\varphi_j(\alpha) = a_j z(\alpha), \quad \psi_j(\alpha) = b_j z(\alpha) \quad (j = 1, 2, 3, 4) \qquad (4.3)$$

$$z(\alpha) = \begin{cases} \sin m\alpha & \text{for odd } j \\ \cos m\alpha & \text{for even } j \end{cases}$$

After (4.3) has been substituted into the system (2.2) and (3.3), the following equation is obtained:

$$\sum_{i=1}^{n}\Big\{\Big[mC_i - \rho_0^4 + \lambda^2\Big(\frac{C_i^3}{3} - \frac{C_i^2}{2} + \frac{C_i}{4} - \frac{1}{12}\Big) - \lambda\Big(B_i^2 + \frac{1}{2}u_i\Big) - m(u_i + C_i^2 - C_i)\Big](\alpha_{i+1} - \alpha_i) + \Big[m^4B_i + \lambda^2\Big(C_i^2B_i - B_iC_i + \frac{B_i}{4}\Big) + \lambda(4A_iB_i + B_iu_i) - m^2(2B_iC_i - B_i)\Big]\Big[\frac{1}{2}(\alpha_{i+1}^2 - \alpha_i^2) - l_1^{i+1} + l_1^i\Big] + \Big[m^4A_i + \lambda^2\Big(A_iC_i^2 - A_iC_i + C_iB_i^2 - \frac{1}{2}B_i^2 + \frac{1}{2}A_i\Big) - \lambda(4A_i^2 + A_iu_i) - m^2\lambda(B_i^2 + 2A_iC_i - A_i)\Big]\Big[\frac{1}{3}(\alpha_{i+1}^3 - \alpha_i^3) - l_2^{i+1} + l_2^i\Big] + \Big[\lambda^2\Big(\frac{2}{3}A_iB_i + \frac{1}{3}B_i^3 + \frac{2}{3}A_iB_iC_i - A_iB_i\Big) - 2m^2\lambda A_iB_i\Big]\Big[\frac{1}{4}(\alpha_{i+1}^4 - \alpha_i^4) - l_3^{i+1} + l_3^i\Big] + \Big[\lambda^2\Big(A_iB_i^2 + A_iC_i - \frac{1}{2}A_i^2\Big) - m^2\lambda A_i^2\Big]\Big[\frac{1}{5}(\alpha_{i+1}^5 - \alpha_i^5) - l_4^{i+1} + l_4^i\Big] + \lambda^2A_iB_i\Big[\frac{1}{6}(\alpha_{i+1}^6 - \alpha_i^6) - l_5^{i+1} + l_5^i\Big] + \frac{\lambda^2}{3}A_i^3\Big[\frac{1}{7}(\alpha_{i+1}^7 - \alpha_i^7) - l_6^{i+1} + l_6^i\Big] + 4\lambda A_i^2m(l_7^{i+1} - l_7^i) + 6\lambda A_iB_im(l_8^{i+1} - l_8^i) + [2\lambda(B_i^2 + 2A_iC_i - 2A_i)m - 4A_im^3](l^{i+1} - l_9^i)\Big\} = 0 \qquad (4.4)$$

Here

$$\rho_0^4 = \frac{qR}{\delta_0E}(1 - \mu^2)k^4(k^2 - 1) - \frac{\delta_0^2}{12R^2}k^4(k^2 - 1)^2$$

$$l_1^i = \frac{1}{4m^2}\cos 2m\alpha_i + \frac{1}{2m}\alpha_i \sin 2m\alpha_i$$

$$l_2^i = \frac{1}{2m^2}\alpha_i \cos 2m\alpha_i + \left(\frac{1}{2m}\alpha_i^2 - \frac{1}{8m^3}\right)\sin 2m\alpha_i$$

$$l_3^i = \left(\frac{3}{4m^2}\alpha_i^2 - \frac{1}{16m^4}\right)\cos 2m\alpha_i + \left(\frac{1}{2m}\alpha_i^3 - \frac{6}{8m^3}\alpha_i\right)\sin 2m\alpha_i$$

$$l_4^i = \frac{1}{2m}\alpha_i^4 \sin 2m\alpha_i - \frac{2}{m}\left\{\left[\frac{3}{4m^2}\alpha_i^2 - \frac{6}{16m^4}\right]\sin 2m\alpha_i - \right.$$
$$\left. - \left(\frac{1}{2m} - \frac{1}{8m^3}\alpha_i\right)\cos 2m\alpha_i\right\}$$

$$l_5^i = \frac{1}{2m}\alpha_i^5 \sin 2m\alpha_i - \frac{5}{2m}\left\{-\frac{1}{2m}\alpha_i^4 \cos 2m\alpha_i + \right.$$
$$\left. + \frac{2}{m}\left(\frac{3}{4m^2}\alpha_i^2 - \frac{1}{16m^4}\right)\cos 2m\alpha_i\right\}$$

$$l_6^i = \frac{1}{2m}\alpha_i^6 \sin 2m\alpha_i - \frac{3}{m}\left\{\frac{1}{2m}\alpha_i^5 \cos 2m\alpha_i + \frac{5}{2m}\left[\frac{1}{2m}\alpha_i^4 \sin 2m\alpha_i - \right.\right.$$
$$\left.\left. - \left(\frac{6}{4m^3}\alpha_i^2 - \frac{6}{8m^5}\right)\sin 2m\alpha_i - \left(\frac{1}{m^2}\alpha_i^3 - \frac{1}{4m^4}\alpha_i\right)\cos 2m\alpha_i\right]\right\}$$

$$l_7^i = \left(\frac{3}{4m^2}\alpha_i^2 - \frac{3}{16m^4}\right)\sin 2m\alpha_i - \left(\frac{1}{2m}\alpha_i^3 - \frac{1}{8m^3}\alpha_i\right)\cos 2m\alpha_i$$

$$l_8^i = \frac{1}{2m^2}\alpha_i \sin 2m\alpha_i - \left(\frac{1}{2m}\alpha_i^2 - \frac{1}{4m^3}\right)\cos 2m\alpha_i$$

$$l_9^i = \frac{1}{4m^2}\sin 2m\alpha_i - \frac{1}{2m}\alpha_i \cos 2m\alpha_i$$

where $A_i^*$, $B_i^*$ and $C_i^*$ are coefficents of the parabola $\varphi_5 = A_i^*\alpha^2 + B_i^*\alpha + C_i^*$ on each section of the constant sign of $u(\alpha)$, and $n$ is the number of control switchings.

The value of $\delta_0$ is determined from the condition that (4.4) equals zero for known $u_0$, $\alpha_i$, $C_1$, $C_2$. It is necessary to determine the auxiliary unknowns $\psi_6$ and $\psi_5$ to determine the sign of $u_0$ and the values of $\alpha_i$.

We obtain as a result of integrating the fifth and sixth equations of the system (3.3):

1. In the presence of a section of the optimal trajectory on the boundary $x_3$ ($\tau_1$, $\tau_2$ are descent and entrance points of the boundary hypersurface)

$$\psi_5 = -\psi_0\alpha - a_1b_4Q_1(\alpha) + A_1$$

$$\psi_6 = \psi_0\frac{\alpha^2}{2} + a_1b_4\int_0^\alpha Q_1(\alpha)\,d\alpha - A_1\alpha + A_2 - a_1b_4Q_2(\alpha)$$

$$(0 \leqslant \alpha \leqslant \tau_1)$$

$$\psi_5 = -\psi_0\alpha - a_1b_4Q_1(\alpha) + A_1 + \varkappa_1$$

$$\psi_6 = \psi_0\frac{\alpha^2}{2} + a_1b_4\int_0^\alpha Q_1(\alpha)\,d\alpha - (A_1 + \varkappa_1)\alpha + \varkappa_1\tau_1 + A_2 - a_1b_4Q_2(\alpha)$$

$$(\tau_1 \leqslant \alpha \leqslant \tau_2)$$

$$\psi_5 = -\psi_0\alpha + A_1 + \varkappa_1 + \varkappa_2 - a_1b_4Q_1(\alpha)$$

$$\psi_6 = \psi_0\frac{\alpha^2}{2} - (A_1 + \varkappa_1 + \varkappa_2)\alpha + \varkappa_1\tau_1 + \varkappa_2\tau_2 + a_1b_4\left[Q_2(\alpha) + \int_0^\alpha Q_1(\alpha)\,d\alpha\right] + A_2$$

$$\left(\tau_2 \leqslant \alpha \leqslant \frac{L}{R}\right)$$
$$A_1 = \psi_5(0), \qquad A_2 = \psi_6(0)$$
$$Q_1(\alpha) = \int_0^\alpha \frac{\psi_4}{\varphi_5^2 a_1 b_4} [L_1(\alpha)\varphi_5 - \Sigma F_j(\alpha)]\, d\alpha$$
$$Q_2(\alpha) = \int_0^\alpha \frac{\psi_4}{\varphi_5 a_1 b_4} L_2(\alpha)\, d\alpha$$

2. When the optimum is realized on trajectories not reaching the boundary $x_3$

$$\psi_5 = -\psi_0\alpha - a_1 b_4 Q_1(\alpha) + A_1^{\cdot}$$
$$\psi_6 = \psi_0 \frac{\alpha^2}{2} + a_1 b_4 \int_0^\alpha Q_1(\alpha)\, d\alpha - a_1 b_4 Q_2(\alpha) - A_1\alpha + A_2$$

The problem of seeking the optimal thickness law $\delta(\alpha)$ reduces to determining the unknown constants $C_1,\ C_2,\ A_1,\ A_2,\ \varkappa_1^{\cdot},\ \varkappa_2,\ \tau_1,\ \tau_2,\ \psi_0,\ u_0\ \ \alpha_i\ (i = 1,2,\ \ldots,\ n)$.

**5.** Two cases are possible in the determination of the unknown constants:

1. *The optimal trajectory emerges on the boundary* $x_3$. *In this case it is necessary* to determine $10 + n$ constants. To do this we have the $10 + n$ conditions

$$\begin{gathered}\varphi_5(0) = 1, \qquad \psi_6(0) = 0, \qquad \varphi_5(\tau_0, \tau_1) = \delta_{\min} / \delta_0 \\ \varphi_6(\tau_0, \tau_1) = 0, \qquad \psi_5(L/R) = 0, \qquad \psi_6(L/R) = 0 \\ \varphi_5(\tau_2, \tau_i) = \delta_{\min} / \delta_0, \qquad \varphi_6(\tau_2, \tau_i) = 0 \\ \psi_0 \leqslant 0, \qquad u_0 = \gamma \operatorname{sign} K(\alpha, \varphi) \qquad (0 \leqslant \alpha \leqslant \alpha_1) \\ K(\alpha_i, \varphi) = 0 \qquad (i = 1, 2, \ldots, n)\end{gathered} \tag{5.1}$$

Moreover, $u \geqslant 0$ on the section $[\tau_0,\ \tau_1],\ [\tau_2,\ \tau_i]$.

2. The optimal trajectory has no sections lying on the boundary hypersurface $x_3$. Then the Pontriagin maximum principle [2] is valid for a problem with moving endpoints and fixed time $(\alpha_1 = L / R)$. The number of constants diminishes to $6 + n$

$$C_1, C_2, A_1, A_2, \psi_0, u_0, \alpha_i \qquad (i = 1, 2, \ldots, n)$$

The boundary conditions are

$$\begin{gathered}\varphi_5(0) = 1, \quad \psi_6(0) = 0, \quad \psi_5(L/R) = 0, \quad \psi_6(L/R) = 0 \\ \psi_0 \leqslant 0, \qquad u_0 = \gamma \operatorname{sign} K(\alpha, \varphi) \qquad (0 \leqslant \alpha \leqslant \alpha_1) \\ K(\alpha_i, \varphi) = 0 \qquad (i = 1, 2, \ldots, n)\end{gathered} \tag{5.2}$$

As computations show, for uniform external pressure the optimum is reached on trajectories emerging on the boundary only for $\delta_{\min} / \delta_0 = 1$. For $\delta_{\min} / \delta_0 < 1$ the case 2 holds. Hence, just two appropriate solutions may be considered.

Since $\varphi_5(0) = 1$ for $\delta_{\min} / \delta_0 = 1$, then naturally $\tau_0 = 0$. This alters the boundary conditions somewhat. The condition $\psi_6(0) = 0$ is necessary in place of $\varphi_6(0) = 0$. Two modifications are hence possible.

i). The quantity $K(0+, \varphi) > 0$, then $u \geqslant 0$ on the portion $[\tau_0,\ \tau_1]$ because of condition (3.9), and hence $\gamma \operatorname{sign} K(\alpha, \varphi) = u_0 > 0$, and $\tau_1 = 0$. Because of the symmetry of the load and the geometry $\tau_2 = \tau_i = L / R$. The $C_1$ and $C_2$ are determined from the conditions $\varphi_5(0) = 1$ and $\varphi_6(0) = 0$, and the expression for $\varphi_5(\alpha)$

becomes

$$\varphi_5(\alpha) = \frac{1}{2}(-1)^i u_0\alpha^2 + 2u_0 \sum_{p=1}^{i} [\alpha_p(-1)^{p+1}\alpha + u_0\alpha_p^2(-1)^i] + 1 \quad (0 \leqslant \alpha \leqslant L/R)$$

From the condition

$$\psi_5(L/R) = 0, \quad A_1 = -\psi_0 L/R + a_1 b_4 Q_1(L/R)$$

and from the condition

$$u_0 = \gamma \operatorname{sign} K(\alpha, \varphi) \qquad (0 \leqslant \alpha \leqslant \alpha_1)$$

and the expression for $K(\alpha, \varphi)$ we obtain $-\operatorname{sign} A_1 = u_0/\gamma$ , or taking into account the homogeneity of the function $H$ relative to $\psi$, we can set

$$A_1 = -\frac{u_0}{\gamma} a_1 b_4$$

The arbitrary constant $A_2$ is determined from the condition $\psi_6(L/R) = 0$

$$A_2 = a_1 b_4 \left[ Q_2\left(\frac{L}{R}\right) - \frac{u_0 L}{\gamma R} - \psi_0 \frac{\alpha^2}{2R^2} - \int_0^{L/R} Q_1(\alpha)\, d\alpha \right]$$

Finally, the conditions

$$K(\alpha_i, \varphi) = 0 \qquad (i = 1, 2, \ldots, n)$$

result in a system of $n$ equations which in combination with the condition

$$\psi_0 = \frac{R}{L} a_1 b_4 \left[ Q_1\left(\frac{L}{R}\right) + \frac{u_0}{\gamma} \right] \leqslant 0$$

should be satisfied by the optimal values of the constants $\alpha_i$, the number of switchings $n$ and the sign of $u_0$.

ii). The quantity $K(0^+, \varphi) < 0$. Then $u_0 = 0$ because of the condition $u \geqslant 0$ on the section $[\tau_0, \tau_1)$. Therefore $\tau_0 = 0$, and $\tau_1$ should be defined. The expressions for $\varphi_5(\alpha)$ are

$$\varphi_5(\alpha) = 1 \qquad (0 \leqslant \alpha \leqslant \tau_1)$$

$$\varphi_5(\alpha) = {}^1/_2 u_0 (\alpha - \tau_1)^2 + C_1(\alpha - \tau_1) + C_2 \qquad (\tau_1 \leqslant \alpha \leqslant \tau_2)$$

Further, $C_1 = 0$ and $C_2 = 1$ from the conditions $\varphi_5(\tau_1) = 1$ and $\varphi_6(\tau_1) = 0$. Since $u(\tau_1) = u_0 > 0$, then $\tau_1$ is defined from the condition $K(\tau_1, \varphi) = 0$. From the condition $\psi_6(0) = 0$ we determine $A_2 = 0$. The condition $u_0 = -\operatorname{sign} A_1\gamma$ is not satisfied on the section $0 \leqslant \alpha \leqslant \tau_1$ , hence, in this case $\psi_0 = -a_1 b_4 < 0$ is assumed because of the homogeneity of $H(\alpha, \varphi, \psi, u)$ in $\psi$, and we determine $A_1$ from the condition $\psi_5(L/R) = 0$. The remaining conditions

$$\varphi_5(\tau_2) = 1, \ \varphi_6(\tau_2) = 0, \ \psi_6(L/R) = 0$$

as well as $K(\alpha_i, \varphi) = 0 \ (i = 1, 2, \ldots, n)$ are used to determine the constants $\varkappa_1$, $\varkappa_2$ and all the $\alpha_i$. In the case $\delta_{min}/\delta_0 < 1$ the determination of the integration constants is analogous to the case $\delta_{min}/\delta_0 = 1$, $K(\alpha, \varphi) > 0$. The exception is the condition $\varphi_6(0) = 0$ which is replaced by the condition $\psi_6(0) = 0$.

The dependences obtained to determine the unknown constants in the general case are a system of nonlinear algebraic equations whose solution is awkward even in the simplest case of $n = 1$. Hence, an algorithm to calculate the constants on a digital computer was constructed for the solution. The program has three branches corresponding to the cases

1) $\delta_{min}/\delta_0 = 1, \quad K(0, \varphi) > 0$
2) $\delta_{min}/\delta_0 = 1, \quad K(0, \varphi) < 0$

3) $\delta_{min}/\delta_0 < 1$

Starting with $\delta = \delta_{min}$, the computation is carried out by means of the dependences of case 1. A standard program to solve nonlinear algebraic equations is used to determine the $\alpha_i$. The integrals in the expressions for $\psi_5(\alpha)$ and $\psi_8(\alpha)$ are evaluated numerically also by using a standard program. The values obtained for the constants are substituted into (4.4) for verification. Upon noncompliance with condition i, the computation is carried out by condition ii, if (4.4) is not zero, then the addition $\delta_{min} + \Delta\delta$ occurs and the computation starts from the beginning for case 3.

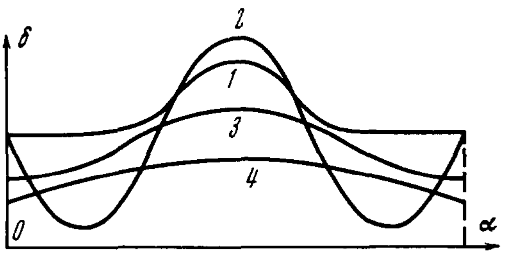


Fig. 1

During the computation, the number $n$ is determined for each $\delta_0$ by comparing several modifications of the computation. Presented in Fig. 1 are the results of computing the thickness distribution for a cylindrical shell with $L/R = 2$ taking account of the constraint $x_3 \leqslant 0$ and without it. Corresponding to curves *1–4* are

$$1 - \quad q^0 = \frac{q}{E}\left(\frac{R}{\delta_{min}}\right) = 0.035, \quad \delta_{min} = 0.08, \quad x_3 \leqslant 0,$$

$$2 - \quad q^0 = 0.035, \quad \delta_{min} = 0.08,$$

$$3 - \quad q^0 = 0.035, \quad \delta_{min} = 0.05,$$

$$4 - \quad q^0 = 0.02, \quad \delta_{min} = 0.03.$$

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